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EXCELLENCE PROGRAM - SYJC (SCIENCE), 2019-2020

SYNOPSIS

MATHEMATICS & STATISTICS – PART 2

INTEGRATION [7 MARKS FOR H.S.C.]

Definition:

If 'f' and 'g' are two functions such that $\frac{d}{dx}[g(x)] = f(x)$, then $g(x)$ is called as anti-derivative or primitive or indefinite integral or simply an integral of $f(x)$ w.r.t. x and we write it as

$$\int f(x)dx = g(x).$$

The process of finding an indefinite integral of a given function is called integration.

THEOREM: If f and g are integrable functions of x then prove that

(a) $\int[f(x)+g(x)]dx = \int f(x)dx + \int g(x)dx$

(b) $\int kf(x)dx = k.\int f(x)dx$ · where k is a constant.

(c) $\int[k_1 f(x) + k_2 g(x)]dx = k_1 \int f(x)dx + k_2 \int g(x)dx.$

where k_1 and k_2 are constant.

Some Important Formulae:

$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n$ $\therefore$ $\int x^n dx = \frac{x^{n+1}}{n+1} + C$ $[n \neq -1]$

$\frac{d}{dx}(\log x) = \frac{1}{x}$ $(x > 0)$ $\therefore$ $\int \frac{1}{x}dx = \log|x| + C$, $x \neq 0$

$\frac{d}{dx}(\sin x) = \cos x$ $\therefore$ $\int \cos x dx = \sin x + C$

$\frac{d}{dx}(\cos x) = -\sin x$ $\therefore$ $\int \sin x dx = -\cos x + C$

$\frac{d}{dx}(\tan x) = \sec^2 x$ $\therefore$ $\int \sec^2 x dx = \tan x + C$

$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$ $\therefore$ $\int \operatorname{cosec}^2 x dx = -\cot x + C$

$\frac{d}{dx}(\sec x) = \sec x \tan x$ $\therefore$ $\int \sec x \tan x dx = \sec x + C$

$\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$ $\therefore$ $\int \operatorname{cosec} x \cot x dx = -\operatorname{cosec} x + C$

$\frac{d}{dx}(e^x) = e^x$ $\therefore$ $\int e^x dx = e^x + C$

$\frac{d}{dx}(a^x) = a^x \log a$ $\therefore \int a^x dx = \frac{a^x}{\log a} + C$ where $C = \frac{C'}{\log a}$

$\frac{d}{dx}(sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$ $\therefore \int \frac{1}{\sqrt{1-x^2}} dx = sin^{-1}x + C$

$\frac{d}{dx}(cos^{-1}x) = \frac{-1}{\sqrt{1-x^2}}$ $= -\cos^{-1}x + C$

$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$ $\therefore \int \frac{1}{1+x^2} dx = \tan^{-1}x + C$

$\frac{d}{dx}(cot^{-1}x) = \frac{-1}{1+x^2}$ $= -\cot^{-1}x + C$

$\frac{d}{dx}(sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$ $\therefore \int \frac{1}{x\sqrt{x^2-1}} dx = sec^{-1}x + C$

$\frac{d}{dx}(cosec^{-1}x) = \frac{-1}{x\sqrt{x^2-1}}$ $= -cosec^{-1}x + C$

THEOREM : If $\int f(x)dx = g(x) + C$ **then**

$$\int f(ax+b)dx = \frac{1}{a}g(ax+b) + C$$ **where** $a \neq 0$ **and** b **is constant.**

NOTE :

The above result shows that, if in any formula on indefinite integrals we change x to $ax + b$ in the integrand, then in the R.H.S. also we change x to $ax + b$ and multiply the R.H.S. by $\frac{1}{a}$.

FORMULAE :

1. (a) $\int x^n dx = \frac{x^{n+1}}{n+1} + C\,(n \neq -1)$ (b) $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C \quad (n \neq -1)$
2. (a) $\int \frac{1}{x} dx = \log|x| + C\,(x \neq 0)$ (b) $\int \frac{1}{ax+b} dx = \frac{1}{a}\log|ax+b| + C$
3. (a) $\int sin\,x\,dx = -cos\,x + C$ (b) $\int sin(ax+b)dx = \frac{-1}{a}cos(ax+b) + C$
4. (a) $\int cos\,x\,dx = sin\,x + C$ (b) $\int cos\,(ax+b)\,dx = \frac{1}{a}sin\,(ax+b) + C$
5. (a) $\int sec^2 x dx = \tan x + C$ (b) $\int \sec^2(ax+b)dx = \frac{1}{a}\tan(ax+b) + C$
6. (a) $\int cosec^2\,x\,dx = -cot\,x + C$ (b) $\int cosec^2(ax+b)\,dx = \frac{-1}{a}cot\,(ax+b) + C$

7. (a) $\int \sec x \tan x dx = \sec x + C$ (b) $\int \sec(ax+b)\tan(ax+b)dx = \frac{1}{a}\sec(ax+b)+C$

8. (a) $\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + C$

(b) $\int \operatorname{cosec}(ax+b)\cot(ax+b)\,dx = \frac{-1}{a}\operatorname{cosec}(ax+b)+C$

9. (a) $\int e^x dx = e^x + C$ (b) $\int e^{ax+b}dx = \frac{1}{a}e^{ax+b}+C$

10. (a) $\int a^x dx = \frac{a^x}{\log a}+C$ (b) $\int a^{px+q}dx = \frac{1}{p}.\frac{a^{px+q}}{\log a}+C$

Integrals of the type $\int \frac{p(x)}{ax+b}dx$ where $p(x)$ is a polynomial in x of degree greater than or equal to 1 and a, b are constants. In such cases, actually divide $p(x)$ by $ax+b$.

For eg.: $\int \frac{5x+4}{2x+1}dx$, $\int \frac{5x^2-6x+3}{2x-3}dx$

Integrals of the type $\int (px+q)\sqrt{ax+b}dx$ **and** $\int \frac{px+q}{\sqrt{ax+b}}dx$.

For eg.: $\int \frac{3x+5}{\sqrt{2x-3}}dx$, $\int x\sqrt{2x-3}dx$

SIMPLE TRIGONOMETRIC FUNCTION

The following formulae from trigonometry will be used in Simplifying trigonometric functions.

1. $\sin^2 x = \frac{1}{2}(1-\cos 2x)$
2. $\cos^2 x = \frac{1}{2}(1+\cos 2x)$
3. $\tan^2 x = \sec^2 x - 1$
4. $\cot^2 x = \operatorname{cosec}^2 x - 1$
5. $\sin A \cos B = \frac{1}{2}[\sin(A+B)+\sin(A-B)]$
6. $\cos A \sin B = \frac{1}{2}[\sin(A+B)-\sin(A-B)]$
7. $\cos A \cos B = \frac{1}{2}[\cos(A+B)+\cos(A-B)]$
8. $\sin A \sin B = \frac{1}{2}[\cos(A-B)-\cos(A+B)]$

9. $sin^3x = \frac{1}{4}(3sinx - sin3x)$

10. $cos^3x = \frac{1}{4}(3cosx + cos3x)$

11. $\sqrt{1+sinx} = cos\frac{x}{2} + sin\frac{x}{2}$

12. $\sqrt{1-sinx} = cos\frac{x}{2} - sin\frac{x}{2}$

13. $\sqrt{1+sin\,2x} = cos\,x + sin\,x$

14. $\sqrt{1-sin\,2x} = cos\,x - sin\,x$

15. $sin\,(x+\frac{\pi}{4}) = sin\,x\,cos\frac{\pi}{4} + cos\,x\,sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}(sinx + cosx)$

$\therefore \quad sin\,x + cos\,x = \sqrt{2}\,sin(x+\frac{\pi}{4})$

16. $cos(x-\frac{\pi}{4}) = cos\,x\,cos\frac{\pi}{4} + sin\,x\,sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}(cos\,x + sin\,x)$

$\therefore \quad cos\,x + sin\,x = \sqrt{2}\,cos\,(x-\frac{\pi}{4}) = \sqrt{2}\,sin\,(x+\frac{\pi}{4})$ ..... [by 15]

17. $sin\,(x-\frac{\pi}{4}) = sin\,x\,cos\frac{\pi}{4} - cos\,x\,sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}(sin\,x - cos\,x)$

$\therefore \quad sin\,x - cos\,x = \sqrt{2}\,sin\,(x-\frac{\pi}{4})$

18. $cos\,(x+\frac{\pi}{4}) = cos\,x\,cos\frac{\pi}{4} - sin\,x\,sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}(cos\,x - sin\,x)$

$\therefore \quad cos\,x - sin\,x = \sqrt{2}\,cos\,(x+\frac{\pi}{4})$

Integration by Substitution :

Consider $\int f(x)dx$. Hence we integrate the function f w. r. t. x, and hence x is the variable of integration. Many times, changing this variable of integration by a suitable substitution, the function to be integrated can be reduced to some standard from. Hence this method is called integration by substitution. In relations to this method we have the following theorem.

THEOREM : If $x = \phi(t)$ is a differentiable function of t then

$$\int f(x)dx = \int f[\phi(t)]\phi'(t)dt$$

Proof : Let $u = \int f(x)dx.$ ......(i)

By definition of primitive,

$$\frac{du}{dx} = f(x) \quad ....(ii)$$

Given $x = \phi(t)$ ....(iii)

$$\frac{dx}{dt} = \phi'(t) \quad ....(iv)$$

From (i) and (iii),

We observe that u is a function of x and x is a function of t.

$\therefore$ u is a composite function of t.

$$\therefore \quad \frac{du}{dt} = \frac{du}{dx}.\frac{dx}{dt}$$

$$\therefore \quad \frac{du}{dt} = f(x)\cdot\frac{dx}{dt} \quad ..... [\text{from (ii)}]$$

$$\therefore \quad \frac{du}{dt} = f[\phi(t)]\phi'(t) \quad ......[\text{from(iii) and (iv)}]$$

By definition of primitive, we have

$$\int f[\phi(t)]\phi'(t)dt = u$$

$$\therefore \quad \int f[\phi(t)]\phi'(t)dt = \int f(x)dx \quad .....[\text{from (i)}]$$

$$\therefore \quad \int f(x)dx = \int f[\phi(t)]\phi'(t)dt$$

NOTE :

If we compare the both sides of this result, then we see that on putting $x = \phi(t)$, the part dx on L.H.S. has been replaced by $\phi'(t)\,dt$. Hence while using the method of substitution we may treat dx and dt as separate quantities and directly write $dx = \phi'(t)\,dt$

Corollaries :

(a) $$\int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1} + C, \quad n \neq -1$$

Proof : Let $I \; = \; \int [f(x)]^n f'(x)dx$

Put $f(x) = t \quad \therefore \quad f'(x)dx = dt$

$$\therefore I \; = \; \int t^n dt = \frac{t^{n+1}}{n+1} + C \quad (n \neq -1)$$

$$= \frac{[f(x)]^{n+1}}{n+1} + C \quad (n \neq -1)$$

(b) $$\int \frac{f'(x)}{f(x)} dx = \log|f(x)| + C$$

Proof : Let $I \; = \; \int \frac{f'(x)}{f(x)} dx$

Put $f(x) = t \quad \therefore \quad f'(x)dx = dt$

$$\therefore I \; = \; \int \frac{dt}{t} = \log|t| + C = \log|f(x)| + C$$

(c) $$\int \frac{f'(x)}{\sqrt{f(x)}} dx = 2\sqrt{f(x)} + C$$

Proof : Let $I \; = \; \int \frac{f'(x)}{\sqrt{f(x)}} dx$

Put $f(x) = t \quad \therefore \quad f'(x)dx = dt$

$$\therefore I \; = \; \int \frac{dt}{\sqrt{t}} = \int t^{-1/2} \; dt = \frac{t^{1/2}}{1/2} + C \; = \; 2\sqrt{t} + C$$

$$= \; 2\sqrt{f(x)} + C$$

(d) $\int f(x)dx = \phi(x)$ then $\int f(ax+b)dx = \frac{1}{a}\phi(ax+b)$

Proof : $\int f(x)dx = \phi(x)$

$$\therefore \quad \int f(t)dt = \phi(t) \qquad \text{....(i)}$$

Let I $= \int f(ax+b)dx$

Put $ax + b = t \quad \therefore \; a\,dx = dt \quad \therefore \; dx = \frac{1}{a} dt$

$$\therefore \quad I = \int f(t)\frac{1}{a}dt = \frac{1}{a}\int f(t)dt$$

$$= \frac{1}{a}\phi(t) \qquad \text{.......[from (i)]}$$

$$= \frac{1}{a}\phi(ax+b)$$

Integrals of the functions : $\tan x$, $\cot x$, $\sec x$, cosec x.

(i) $\int \tan x dx = -\log|cosx| + C = \log|secx| + C$

and $\int \tan(ax+b)dx = \frac{1}{a}\log|sec(ax+b)| + C$

(ii) $\int cotxdx = \log|sinx| + C$

and $\int cot(ax+b)dx = \frac{1}{a}\log|sin(ax+b)| + C$

(iii) $\int secdx = \log|secx + \tan x| + C$

and $\int sec(ax+b)dx = \frac{1}{a}\log|sec(ax+b) + \tan(ax+b)| + C$

NOTE : $sec\, x + \tan x = \tan(\frac{x}{2}+\frac{\pi}{4})$

$$\therefore \quad \int secxdx = \log\left|\tan(\frac{x}{2}+\frac{\pi}{4})\right| + C$$

(iv) $\int co\sec xdx = \log|co\sec x - cotx| + C$

and $\int co\sec(ax+b)dx = \frac{1}{a}\log|co\sec(ax+b) - cot(ax+b)| + C$

NOTE : $co\sec x - cotx = \tan\frac{x}{2}$

$$\therefore \quad \int co\sec xdx = \log\left|\tan\frac{x}{2}\right| + C$$

Integrals of the type : $\int \frac{a\, sinx + b\, cosx}{c\, sinx + d\, cosx}dx$

Method : The numerator is to be split up in two parts as

A(denominator)+B$\left[\frac{d}{dx}\text{(denominator)}\right]$ where A and B are constants which are to be found by equating coefficients of sin$x$ and cos$x$

Integrals of the type : $\int \frac{ae^x + b}{ce^x + d} dx$

Method : Express $ae^x + b = A\left(ce^x + d\right) + B\frac{d}{dx}\left(ce^x + d\right)$ where A and B are constants which are to be found.

If the functions to be integrated contain expressions contain expressions of the form given below, then it is convenient to use trigonometric substations as given below:

If integrand contains	Substitution
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$ or $x = a\cos\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$ or $x = a\operatorname{cosec}\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$ or $x = a\cot\theta$
$\sqrt{\frac{a-x}{a+x}}$	$x = a\cos\theta$
$\sqrt{\frac{a-x}{x}}$	$x = a\sin^2\theta$
$\sqrt{2ax - x^2}$	$x = 2a\sin^2\theta$

Some Standard Formulae :

(i) $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + C$

(ii) $\int \frac{1}{\sqrt{x^2 - a^2}} dx = \log\left|x + \sqrt{x^2 - a^2}\right| + C$

(iii) $\int \frac{dx}{\sqrt{x^2 + a^2}} = \log\left|x + \sqrt{x^2 + a^2}\right| + C$

(iv) $\int \frac{1}{x^2 + a^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

(v) $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$

(vi) $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$

Integrals of the type : $\int \frac{1}{ax^2+bx+c}dx$, $\int \frac{1}{\sqrt{ax^2+bx+c}}dx$

In such cases we express ax^2+bx+c as the sum or difference of two squares. (Completing the square method)

Integrals of the type : $\int \frac{px+q}{ax^2+bx+c}dx$

Method : Express $px+q$ as $A\left[\frac{d}{dx}(ax^2+bx+c)\right]+B$ where A & B are constants to be determined by equating the Coefficients of various power of x.

Integrals of the type : $\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$

Method : Express $px+q=A\left[\frac{d}{dx}\left(ax^2+bx+c\right)\right]+B$ where A & B are constants to be determined by equating the coefficients of various powers of x.

Integrals of the type : $\int \frac{dx}{a+b\sin x}$, $\int \frac{dx}{a+b\cos x}$, $\int \frac{dx}{a\sin x+b\cos x+c}$

Method : Put $\tan\frac{x}{2}=t$ $\qquad \therefore \frac{x}{2}=\tan^{-1}t$

$\therefore x=2\tan^{-1}t \qquad \therefore dx=\frac{2dt}{1+t^2}$

$$\sin x=\frac{2\tan x/2}{1+\tan^2 x/2}=\frac{2t}{1+t^2}$$

$$\cos x=\frac{1-\tan^2 x/2}{1+\tan^2 x/2}=\frac{1-t^2}{1+t^2}$$

Integrals of the type : $\int \frac{dx}{a+b\sin^2 x}$, $\int \frac{dx}{a+b\cos^2 x}$, $\int \frac{dx}{a\sin^2 x+b\cos^2 x}$

Method : Divide numerator and denominator by $\cos^2 x$ and put $\tan x=t$

INTEGRATION BY PARTS

Theorem :If u is a differentiable function of x and v is an integrable function of x then

$$\int uv\,dx = u\int v\,dx - \int\left(\frac{du}{dx}\int v\,dx\right)dx.$$

Proof : Let $\int v\,dx = w$ ...(i)

$\therefore$ By definition of primitive, we have

$$\frac{dw}{dx} = v \qquad ...(ii)$$

We know that $\frac{d}{dx}(uw) = u\frac{dw}{dx} + w\frac{du}{dx}$

$\therefore$ By definition of primitive, we have

$$\int\left[u\frac{dw}{dx} + w\frac{du}{dx}\right]dx = uw$$

$\therefore$ $\int u\frac{dw}{dx}dx + \int w\frac{du}{dx}dx = uw$ $\therefore$ $\int u\frac{dw}{dx}dx = uw - \int w\frac{du}{dx}dx$

Eliminating w using (i) and (ii) , we get

$$\int uv\,dx = u\int v\,dx - \int\left(\frac{du}{dx}\int v\,dx\right)dx$$

An alternative form of the above theorem :

$$\int f(x).g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

Proof : We know that

$$\frac{d}{dx}\left[f(x).g(x)\right] = f(x)g'(x) + f'(x).g(x).$$

By definition of primitive, we have

$$\int\left[f(x).g'(x) + f'(x).g(x)\right]dx = f(x).g(x)$$

$$\therefore \quad \int f(x)g'(x)dx + \int f'(x)g(x)dx = f(x)g(x)$$

$$\therefore \quad \int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

NOTE :

I. (a) If in the above rule of integration by parts we put f(x) = u and g(x) = v so that

$g'(x) = \frac{dv}{dx}$ then the rule take the form,

$$\int\left(u.\frac{dv}{dx}\right)dx = uv - \int\left(\frac{du}{dx}\right)vdx$$

(b) If we put $f(x) = u$ and $g'(x) = v$ so that $g(x) = \int vdx$ then the rule takes the form,

$$\int uvdx = u\int vdx - \int\left[\frac{du}{dx}\int vdx\right]dx$$

II. Since the rule of integration by parts is to be used when the integrand is a product of two functions, out of which second can be easily integrated, we have to make a proper choice of the first function and the second function. Let the Algebraic, Trignometric, Inverse, Exponential and Logarithmic functions be denoted by A,T, I, E & L respectively. The first function should be the one which comes first in the order L I A T E .

Formula: $\int e^x\left[f(x)+f'(x)\right]dx = e^x f(x)+C$

Integrals of type : $\sqrt{a^2-x^2}$, $\sqrt{x^2+a^2}$, $\sqrt{x^2-a^2}$

(i) $\int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}sin^{-1}\left(\frac{x}{a}\right)+C$

(ii) $\int\sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right|+C$

(iii) $\int\sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+C$

INTEGRALS OF THE TYPE : $\int(px+q)\sqrt{ax^2+bx+c}dx$

Express px + q in the form $A\frac{d}{dx}\left(ax^2+bx+c\right)+B$ **where A and B are constants to be determined by equating various powers of x.**

INTEGRATION BY PARTIAL FRACTIONS :

The following table indicates how to choose the partial fractions corresponding to the given rational function in different forms.

Sr. No.	Form of the rational function	Form of the partial fraction
1.	$\frac{px+q}{(x-a)(x-b)}$, $a \neq b$.	$\frac{A}{x-a}+\frac{B}{x-b}$
2.	$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$, a,b,c *are distinct*	$\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}$
3.	$\frac{px+q}{(x-a)^2}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}$
4.	$\frac{px^2+qx+r}{(x-a)^3}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}+\frac{C}{(x-a)^3}$
5.	$\frac{px^2+qx+r}{(x-a)^2(x-b)}$, $a \neq b$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}+\frac{C}{(x-b)}$
6.	$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$, where $x^2 + bx + c$ cannot be factorized	$\frac{A}{x-a}+\frac{Bx+C}{x^2+bx+c}$